

Baxter's Q-operator for the W -algebra W_N

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 355206

(<http://iopscience.iop.org/1751-8121/41/35/355206>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.150

The article was downloaded on 03/06/2010 at 07:08

Please note that [terms and conditions apply](#).

Baxter's Q -operator for the W -algebra W_N

Takeo Kojima

Department of Mathematics, College of Science and Technology, Nihon University, Surugadai,
Chiyoda-ku, Tokyo 101-0062, Japan

Received 30 March 2008, in final form 29 June 2008

Published 25 July 2008

Online at stacks.iop.org/JPhysA/41/355206

Abstract

The q -oscillator representation for the Borel subalgebra of the affine symmetry $U'_q(\widehat{sl}_N)$ is presented. By means of this q -oscillator representation, we give the free field realizations of Baxter's Q -operator $\mathbf{Q}_j(\lambda)$, $\overline{\mathbf{Q}}_j(\lambda)$, ($j = 1, 2, \dots, N$) for the W -algebra W_N . We give functional relations of the T - Q operators, including the higher-rank generalization of Baxter's T - Q relation.

PACS number: 43.25.

Mathematics Subject Classification: 81U15, 81R10

1. Introduction

Baxter's T - Q operators have various exceptional properties, and play an important role in many aspects of the theory of integrable systems. Originally the Q -operator was introduced by Baxter [1], in terms of some special transfer matrix of the eight-vertex model. Over the last three decades, this method of the Q -operator has been developed in many literatures. We would like to refer to some of these literatures, written by Baxter [2–5], Takhtadzhian and Faddeev [6], Fabricius and McCoy [7–9], Fabricius [10], Bazhanov and Mangazeev [11], Feigin *et al* [12], and Kojima and Shiraishi [13]. However a full theory of the Q -operator for the eight-vertex model is not yet developed. For the simpler models associated with the quantum group $U_q(g)$, there have been many papers which extend, generalize, and comment on the T - Q relation. We would like to refer to some of these literatures, including Sklyanin's separation variable method, written by Sklyanin [14–16], Kuznetsov *et al* [17], Pasquier and Gaudin [18], Derkachov [19], Derkachov *et al* [20–22], Derkachov and Mansahov [23], Belisty *et al* [24], Korff [25, 26], Bytsko and Teschner [27], Bazhanov *et al* [28–31, 34], Rossi and Weston [32], Dorey and Tateo [33], Kulish and Zeitlin [35], Antonov and Feigin [36], Krichever *et al* [38], Bazhanov and Reshetikhin [39], Kuniba *et al* [40], Boos *et al* [41, 42], and Chervov and Falqui [43]. Each paper added to our understanding of the great Baxter's original paper [1]. Especially, for example, the T - Q operators acting on the Fock space of the Virasoro algebra Vir were introduced by Bazhanov, Lukyanov and Zamolodchikov [28–30]. They derived various functional relations of the T - Q operators and gave the asymptotic behaviour of the eigenvalue of the T - Q operators. Dorey and Tateo

[33] revealed the hidden connection between the vacuum expectation value of the Q -operator and the spectral determinant for Schrödinger equation. Bazhanov *et al* [34] achieved the W_3 -algebraic generalization of [28, 29, 30, 31, 33]. In this paper we study the higher-rank W_N -generalization of [34]. We study the T - Q operators acting on the Fock space of the W -algebra W_N . We give the free field realization of the Q -operator and functional relations of the T - Q operators for the W -algebra W_N , including the higher-rank generalization of Baxter's T - Q relation,

$$\begin{aligned} \mathbf{Q}_i(tq^N) + \sum_{s=1}^{N-1} (-1)^s \mathbf{T}_{\Lambda_1+\dots+\Lambda_s}(tq^{-1}) \mathbf{Q}_i(tq^{N-2s}) + (-1)^N \mathbf{Q}_i(tq^{-N}) &= 0, \\ \overline{\mathbf{Q}}_i(tq^{-N}) + \sum_{s=1}^{N-1} (-1)^s \overline{\mathbf{T}}_{\Lambda_1+\dots+\Lambda_s}(tq) \overline{\mathbf{Q}}_i(tq^{-N+2s}) + (-1)^N \overline{\mathbf{Q}}_i(tq^N) &= 0, \end{aligned}$$

where $i = 1, 2, \dots, N$. The organization of this paper is as follows. In section 2, we give basic definitions, including q -oscillator representation of the Borel subalgebra of the affine symmetry $U'_q(\widehat{sl}_N)$, which plays an essential role in the construction of the Q -operator. In section 3, we give the definitions of the T - and Q -operators. In section 4, we give conjecturous functional relations between the T - and Q -operators, including Baxter's T - Q relation. In the appendix, we give supporting arguments on conjecturous formulae stated in section 4.

2. Basic definition

In this section, we give the different realizations of the Borel subalgebra of the affine quantum algebra $U'_q(\widehat{sl}_N)$, which will play an important role in the construction of Baxter's T - Q operators. Let us fix the integer $N \geq 3$ and a complex number $1 < r < N$. In this paper, upon this setting, we construct Baxter's T - Q operators on the space of the W -algebra W_N with the central charge $-\infty < C_{\text{CFT}} < -2$, where

$$C_{\text{CFT}} = (N - 1) \left(1 - \frac{N(N + 1)}{r(r - 1)} \right).$$

Because $C_{\text{CFT}} \rightarrow -\infty$ represents the classical limit, we call $-\infty < C_{\text{CFT}} < -2$ 'quasi-classical domain'. By analytic continuation, it is possible to extend our theory to the CFT with central charge $C_{\text{CFT}} < 1$. We would like to note that the unitary minimal CFT is described by the central charge $C_{\text{CFT}} = (N - 1) \left(1 - \frac{N(N+1)}{r(r-1)} \right)$ for $N, r \in \mathbb{Z}, (N \geq 2, r \geq N + 2)$ [44]. We set parameters $r^* = r - 1$ and $q = e^{2\pi i \frac{r^*}{r}}$. In what follows we use the q -integer $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

2.1. The q -oscillator representation

Let $\{\epsilon_j\}$ be an orthonormal basis of \mathbb{R}^N , relative to the standard inner product $(\epsilon_i | \epsilon_j) = \delta_{i,j}$. Let us set $\bar{\epsilon}_j = \epsilon_j - \epsilon$, where $\epsilon = \frac{1}{N} \sum_{j=1}^N \epsilon_j$. We have $(\bar{\epsilon}_i | \bar{\epsilon}_j) = \delta_{i,j} - \frac{1}{N}$. Let us set the simple roots $\alpha_j = \bar{\epsilon}_j - \bar{\epsilon}_{j+1}$, ($1 \leq j \leq N - 1$) and $\alpha_N = -\sum_{j=1}^{N-1} \alpha_j$. Let us set the fundamental weights ω_j as the dual vector of α_j , i.e. $(\alpha_i | \omega_j) = \delta_{i,j}$. Explicitly we have $\omega_j = \bar{\epsilon}_1 + \dots + \bar{\epsilon}_j$. Let us set the weight lattice $P = \bigoplus_{j=1}^N \mathbb{Z} \bar{\epsilon}_j$. We consider the quantum affine algebra $U'_q(\widehat{sl}_N)$, which is generated by $e_1, \dots, e_N, f_1, \dots, f_N$, and h_1, \dots, h_N , with the defining relations,

$$\begin{aligned} [h_i, h_j] &= 0, \quad [h_i, e_j] = (\alpha_i | \alpha_j) e_j, \quad [h_i, f_j] = -(\alpha_i | \alpha_j) f_j, \quad [e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \\ e_i^2 e_j - [2]_q e_i e_j e_i + e_j e_i^2 &= 0, \quad f_i^2 f_j - [2]_q f_i f_j f_i + f_j f_i^2 = 0, \quad \text{for } (\alpha_i | \alpha_j) = -1. \end{aligned}$$

Here $((\alpha_j|\alpha_k))_{1 \leq j,k \leq N}$ is the Cartan matrix of type \widehat{sl}_N . Let us introduce the Borel subalgebra of $U'_q(\widehat{sl}_N)$. The Borel subalgebra $U'_q(\widehat{\mathfrak{b}}^+)$ is generated by $e_1, \dots, e_N, h_1, \dots, h_N$, and $U'_q(\widehat{\mathfrak{b}}^-)$ by $f_1, \dots, f_N, h_1, \dots, h_N$. In this paper we consider the level $c = 0$ case, with the central element $c = h_1 + \dots + h_N$. Let us introduce the q -oscillator representation o_t of the Borel subalgebra $U'_q(\widehat{\mathfrak{b}}^+)$. The q -oscillator algebra Osc_j , ($1 \leq j \leq N-1$), is generated by elements $\mathcal{E}_j, \mathcal{E}_j^*, \mathcal{H}_j$, with the defining relations,

$$[\mathcal{H}_j, \mathcal{E}_j] = \mathcal{E}_j, \quad [\mathcal{H}_j, \mathcal{E}_j^*] = -\mathcal{E}_j^*, \quad q\mathcal{E}_j\mathcal{E}_j^* - q^{-1}\mathcal{E}_j^*\mathcal{E}_j = \frac{1}{q - q^{-1}}. \quad (2.1)$$

Let us set $\text{Osc} = \text{Osc}_1 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} \text{Osc}_{N-1}$. We have $[\mathcal{E}_j, \mathcal{E}_k] = 0, [\mathcal{E}_j^*, \mathcal{E}_k^*] = 0, [\mathcal{E}_j, \mathcal{E}_k^*] = 0, [\mathcal{H}_j, \mathcal{H}_k] = 0$ for $j \neq k$. Let us set the auxiliary operator $\mathcal{H}_N = -\mathcal{H}_1 - \mathcal{H}_2 - \dots - \mathcal{H}_{N-1}$. We define homomorphism $o_t : U'_q(\widehat{\mathfrak{b}}^+) \rightarrow \text{Osc}$ by

$$\begin{aligned} o_t(e_1) &= tq^{\frac{1}{2}}(q - q^{-1})q^{-\mathcal{H}_2}\mathcal{E}_1^*\mathcal{E}_2, \\ o_t(e_2) &= q^{\frac{1}{2}}(q - q^{-1})q^{-\mathcal{H}_3}\mathcal{E}_2^*\mathcal{E}_3, \\ &\dots \\ o_t(e_{N-2}) &= q^{\frac{1}{2}}(q - q^{-1})q^{-\mathcal{H}_{N-1}}\mathcal{E}_{N-2}^*\mathcal{E}_{N-1}, \\ o_t(e_{N-1}) &= \mathcal{E}_{N-1}^*, \\ o_t(e_N) &= q^{-\mathcal{H}_1 - \mathcal{H}_N}\mathcal{E}_1, \\ o_t(h_1) &= -\mathcal{H}_1 + \mathcal{H}_2, \quad o_t(h_2) = -\mathcal{H}_2 + \mathcal{H}_3, \dots, o_t(h_N) = -\mathcal{H}_N + \mathcal{H}_1. \end{aligned} \quad (2.2)$$

This q -oscillator representation o_t satisfies the level zero condition $o_t(h_1 + h_2 + \dots + h_N) = 0$. This q -oscillator representation gives a higher-rank generalization of those in [34]. By means of the Dynkin diagram automorphism τ, σ , we construct a family of the q -oscillator representation $o_{t,j}, \bar{o}_{t,j}$. Let us set the Dynkin diagram automorphism τ of the affine algebra $U'_q(\widehat{sl}_N)$,

$$\begin{aligned} \tau(e_1) &= e_2, \dots, \tau(e_j) = e_{j+1}, \dots, \tau(e_N) = e_1, \\ \tau(h_1) &= h_2, \dots, \tau(h_j) = h_{j+1}, \dots, \tau(h_N) = h_1, \\ \tau(f_1) &= f_2, \dots, \tau(f_j) = f_{j+1}, \dots, \tau(f_N) = f_1. \end{aligned}$$

Let us set the Dynkin diagram automorphism σ of the finite simple algebra $U_q(sl_N)$, generated by $e_2, \dots, e_N, h_2, \dots, h_N, f_2, \dots, f_N$.

$$\begin{aligned} \sigma(e_2) &= e_N, \dots, \sigma(e_j) = e_{N+2-j}, \dots, \sigma(e_N) = e_2, \\ \sigma(h_2) &= h_N, \dots, \sigma(h_j) = h_{N+2-j}, \dots, \sigma(h_N) = h_2, \\ \sigma(f_2) &= f_N, \dots, \sigma(f_j) = f_{N+2-j}, \dots, \sigma(f_N) = f_2, \end{aligned}$$

and σ is extended to the affine vertex as $\sigma(e_1) = e_1, \sigma(h_1) = h_1, \sigma(f_1) = f_1$. We have the action of $\tau^j \cdot \sigma \cdot \tau^{-1}$,

$$\begin{aligned} \tau^j \cdot \sigma \cdot \tau^{-1}(e_i) &= e_{j-1-i}, \\ \tau^j \cdot \sigma \cdot \tau^{-1}(h_i) &= h_{j-1+i}, \\ \tau^j \cdot \sigma \cdot \tau^{-1}(f_i) &= f_{j-1-i}, \end{aligned}$$

with $s, j \in \mathbb{Z}$. We set homomorphism $o_{t,j}, \bar{o}_{t,j} : U'_q(\widehat{\mathfrak{b}}^+) \rightarrow \text{Osc}$, ($1 \leq j \leq N$),

$$o_{t,j} = o_t \cdot \tau^{-j}, \quad \bar{o}_{t,j} = o_{(-1)^N t} \cdot \tau^j \cdot \sigma \cdot \tau^{-1}, \quad (2.3)$$

These q -oscillator representations $o_{t,j}, \bar{o}_{t,j}$ will play an important role in the construction of Baxter's Q -operator.

2.2. Evaluation highest weight representation

Let us consider the quantum simple algebra $U_q(gl_N)$, which is generated by $E_{\alpha_1}, \dots, E_{\alpha_{N-1}}, H_1, \dots, H_N$, and $F_{\alpha_1}, \dots, F_{\alpha_{N-1}}$, with the defining relations,

$$[H_i, H_j] = 0, \quad [H_i, E_{\alpha_j}] = (\delta_{i,j} - \delta_{i,j+1})E_{\alpha_j}, \quad [H_i, F_{\alpha_j}] = (-\delta_{i,j} + \delta_{i,j+1})F_{\alpha_j},$$

$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{i,j} \frac{q^{H_i - H_{i+1}} - q^{-H_i + H_{i+1}}}{q - q^{-1}},$$

$$E_{\alpha_i}^2 E_{\alpha_j} - [2]_q E_{\alpha_i} E_{\alpha_j} E_{\alpha_i} + E_{\alpha_j} E_{\alpha_i}^2 = 0, \quad F_{\alpha_i}^2 F_{\alpha_j} - [2]_q F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} + F_{\alpha_j} F_{\alpha_i}^2 = 0.$$

Let us set the root vectors,

$$F_{\alpha_1 + \alpha_2} = [F_{\alpha_2}, F_{\alpha_1}]_{\sqrt{q}} = \sqrt{q} F_{\alpha_2} F_{\alpha_1} - \frac{1}{\sqrt{q}} F_{\alpha_1} F_{\alpha_2},$$

$$\bar{F}_{\alpha_1 + \alpha_2} = [F_{\alpha_2}, F_{\alpha_1}]_{\frac{1}{\sqrt{q}}} = \frac{1}{\sqrt{q}} F_{\alpha_2} F_{\alpha_1} - \sqrt{q} F_{\alpha_1} F_{\alpha_2},$$

$$F_{\alpha_1 + \dots + \alpha_{N-1}} = [F_{\alpha_{N-1}}, [F_{\alpha_{N-2}}, \dots, [F_{\alpha_2}, F_{\alpha_1}]_{\sqrt{q}} \dots]_{\sqrt{q}}]_{\sqrt{q}},$$

$$\bar{F}_{\alpha_1 + \dots + \alpha_{N-1}} = [[\dots, [F_{\alpha_{N-1}}, F_{\alpha_{N-2}}]_{\frac{1}{\sqrt{q}}} \dots, F_{\alpha_2}]_{\frac{1}{\sqrt{q}}}, F_{\alpha_1}]_{\frac{1}{\sqrt{q}}}.$$

Let us set the automorphism σ by

$$\sigma(E_{\alpha_1}) = E_{\alpha_{N-1}}, \dots, \sigma(E_{\alpha_j}) = E_{\alpha_{N-j}}, \dots, \sigma(E_{\alpha_{N-1}}) = E_{\alpha_1},$$

$$\sigma(H_1) = -H_N, \dots, \sigma(H_j) = -H_{N-j+1}, \dots, \sigma(H_N) = -H_1,$$

$$\sigma(F_{\alpha_1}) = F_{\alpha_{N-1}}, \dots, \sigma(F_{\alpha_j}) = F_{\alpha_{N-j}}, \dots, \sigma(F_{\alpha_{N-1}}) = F_{\alpha_1}.$$

We have the evaluation representation $ev_t, \bar{ev}_t: U'_q(\widehat{sl}_N) \rightarrow U_q(gl_N)$, given by

$$ev_t(e_2) = E_{\alpha_1}, \dots, ev_t(e_{j+1}) = E_{\alpha_j}, \dots, ev_t(e_N) = E_{\alpha_{N-1}},$$

$$ev_t(h_2) = H_1 - H_2, \dots, ev_t(h_{j+1}) = H_j - H_{j+1}, \dots, ev_t(h_N) = H_{N-1} - H_N,$$

$$ev_t(f_2) = F_{\alpha_1}, \dots, ev_t(f_{j+1}) = F_{\alpha_j}, \dots, ev_t(f_N) = F_{\alpha_{N-1}},$$

$$ev_t(e_1) = t F_{\alpha_1 + \alpha_2 + \dots + \alpha_{N-1}} q^{H_1 + H_N}, \quad ev_t(f_1) = t^{-1} E_{\alpha_1 + \alpha_2 + \dots + \alpha_{N-1}} q^{-H_1 - H_N},$$

$$ev_t(h_1) = H_N - H_1.$$

$$\bar{ev}_t(e_2) = E_{\alpha_1}, \dots, \bar{ev}_t(e_{j+1}) = E_{\alpha_j}, \dots, \bar{ev}_t(e_N) = E_{\alpha_{N-1}},$$

$$\bar{ev}_t(h_2) = H_1 - H_2, \dots, \bar{ev}_t(h_{j+1}) = H_j - H_{j+1}, \dots, \bar{ev}_t(h_N) = H_{N-1} - H_N,$$

$$\bar{ev}_t(f_2) = F_{\alpha_1}, \dots, \bar{ev}_t(f_{j+1}) = F_{\alpha_j}, \dots, \bar{ev}_t(f_N) = F_{\alpha_{N-1}},$$

$$\bar{ev}_t(e_1) = t \bar{F}_{\alpha_1 + \alpha_2 + \dots + \alpha_{N-1}} q^{-H_1 - H_N}, \quad \bar{ev}_t(f_1) = t^{-1} \bar{E}_{\alpha_1 + \alpha_2 + \dots + \alpha_{N-1}} q^{H_1 + H_N},$$

$$\bar{ev}_t(h_1) = H_N - H_1.$$

We have the conjugation $\bar{ev}_t = \sigma \cdot ev_{(-)^{N_t}} \cdot \sigma^{-1}$. We set the irreducible highest representation of $U_q(gl_N)$ with the highest weight $\lambda = m_1 \Lambda_1 + \dots + m_N \Lambda_N$, the highest weight vector $|\lambda\rangle$ of $U_q(sl_N)$:

$$\pi^{(\lambda)}(E_{\alpha_j})|\lambda\rangle = 0, \quad \pi^{(\lambda)}(H_j)|\lambda\rangle = m_j |\lambda\rangle, \quad (1 \leq j \leq N).$$

In what follows we consider the case $m_j - m_{j+1} \in \mathbb{N}$, ($1 \leq j \leq N - 2$). In this case the representation $\pi^{(\lambda)}$ is finite dimension. Let us set the evaluation highest weight representation $\pi_t^{(\lambda)}$ of the affine symmetry $U'_q(\widehat{sl}_N)$ as

$$\pi_t^{(\lambda)} = \pi^{(\lambda)} \cdot ev_t, \quad \bar{\pi}_t^{(\lambda)} = \pi^{(\lambda)} \cdot \bar{ev}_t.$$

These evaluation highest weight representation will play an important role in the construction of the T -operator $\mathbf{T}_\lambda(t), \bar{\mathbf{T}}_\lambda(t)$.

2.3. Screening current

Let us introduce bosons B_m^i , ($m \in \mathbb{Z}_{\neq 0}$; $i = 1, 2, \dots, N - 1$), by

$$[B_m^i, B_n^j] = m\delta_{m+n,0}(\alpha_i|\alpha_j)\frac{r-1}{r}, \quad (1 \leq i, j \leq N-1). \tag{2.4}$$

Let us set $B_m^N = -\sum_{j=1}^{N-1} B_m^j$. We have the commutation relation $[B_m^i, B_n^j] = m\delta_{m+n,0}(\alpha_i|\alpha_j)\frac{r-1}{r}$, for $1 \leq i, j \leq N$. Let us set the zero-mode operators P_λ and Q_λ , ($\lambda \in P = \bigoplus_j \mathbb{Z}\bar{\epsilon}_j$) by

$$[P_\lambda, iQ_\mu] = (\lambda|\mu). \tag{2.5}$$

Let us set the Heisenberg algebra \mathcal{B} generated by $B_m^1, \dots, B_m^{N-1}, P_\lambda, Q_\lambda$, ($\lambda \in P$) and its completion $\widehat{\mathcal{B}}$. Let us set the Fock space $\mathcal{F}_{l,k}$ by

$$B_m^j|l, k\rangle = 0, \quad (m > 0) \tag{2.6}$$

$$P_\alpha|l, k\rangle = \left(\alpha \left| \sqrt{\frac{r}{r-1}}l - \sqrt{\frac{r-1}{r}}k \right.\right) |l, k\rangle, \tag{2.7}$$

$$|l, k\rangle = e^{i\sqrt{\frac{r}{r-1}}Q_{l-i}\sqrt{\frac{r-1}{r}}Q_k}|0, 0\rangle. \tag{2.8}$$

Let us set the screening currents of the W -algebra W_N by

$$\begin{aligned} V_{\alpha_j}(u) &= \exp\left(i\sqrt{\frac{r^*}{r}}Q_{\alpha_j}\right) \exp\left(\sqrt{\frac{r^*}{r}}P_{\alpha_j}iu\right) \exp\left(\sum_{m>0}\frac{1}{m}B_{-m}^j e^{imu}\right) \\ &\times \exp\left(-\sum_{m>0}\frac{1}{m}B_m^j e^{-imu}\right), \quad (1 \leq j \leq N). \end{aligned} \tag{2.9}$$

Here we have added one operator $V_{\alpha_N}(u)$, which looks like affinization of the classical A_{N-1} . We can find the elliptic deformation of $V_{\alpha_j}(u)$ for $j \neq N$ in [12, 13]. For $\text{Re}(u_1) > \text{Re}(u_2)$, we have

$$\begin{aligned} V_{\alpha_j}(u_1)V_{\alpha_j}(u_2) &=: V_{\alpha_j}(u_1)V_{\alpha_j}(u_2) : (e^{iu_1} - e^{iu_2})^{\frac{2r^*}{r}}, \quad (1 \leq j \leq N), \\ V_{\alpha_j}(u_1)V_{\alpha_{j+1}}(u_2) &=: V_{\alpha_j}(u_1)V_{\alpha_{j+1}}(u_2) : (e^{iu_1} - e^{iu_2})^{-\frac{r^*}{r}}, \quad (1 \leq j \leq N), \\ V_{\alpha_{j+1}}(u_1)V_{\alpha_j}(u_2) &=: V_{\alpha_{j+1}}(u_1)V_{\alpha_j}(u_2) : (e^{iu_1} - e^{iu_2})^{-\frac{r^*}{r}}, \quad (1 \leq j \leq N). \end{aligned}$$

By analytic continuation, we have

$$V_{\alpha_i}(u_1)V_{\alpha_j}(u_2) = q^{(\alpha_i|\alpha_j)}V_{\alpha_j}(u_2)V_{\alpha_i}(u_1), \quad (1 \leq i, j \leq N). \tag{2.10}$$

Let us set

$$z_j = \exp\left(-2\pi i\sqrt{\frac{r^*}{r}}P_{\bar{\epsilon}_j}\right), \quad (1 \leq j \leq N). \tag{2.11}$$

We have $z_1 z_2 \cdots z_N = 1$ and

$$V_{\alpha_i}(u + 2\pi) = z_i^{-1}z_{i+1}V_{\alpha_i}(u), \quad z_i V_{\alpha_j}(u) = q^{\delta_{i,j+1} - \delta_{i,j}}V_{\alpha_j}(u)z_i.$$

Let us set the nilpotent subalgebra $U'_q(\widehat{\mathfrak{n}}^-)$ generated by f_1, f_2, \dots, f_N . We have homomorphism $sc : U'_q(\widehat{\mathfrak{n}}^-) \rightarrow \widehat{\mathcal{B}}$ given by

$$sc(f_j) = \frac{1}{q - q^{-1}} \int_0^{2\pi} V_{\alpha_j}(u) du, \quad (1 \leq j \leq N).$$

3. Baxter's Q -operator

In this section, we define Baxter's T - Q operators by means of the trace of the universal R , and present conjecturous functional relations of the T - Q operator, which include the higher-rank generalization of Baxter's T - Q relation.

3.1. \mathcal{L} -operator

Let us set the universal L -operator $\mathcal{L} \in \widehat{B} \otimes U_q(\widehat{\mathfrak{n}}^-)$ by

$$\mathcal{L} = \exp\left(-\pi i \sqrt{\frac{r^*}{r}} \sum_{j=1}^N P_{\omega_j} \otimes h_j\right) \mathcal{P} \exp\left(\int_0^{2\pi} K(u) du\right). \quad (3.1)$$

Here we have set

$$K(u) = \sum_{j=1}^N V_{\alpha_j}(u) \otimes e_j.$$

Here $\mathcal{P} \exp\left(\int_0^{2\pi} K(u) du\right)$ represents the path ordered exponential

$$\begin{aligned} \mathcal{P} \exp\left(\int_0^{2\pi} K(u) du\right) &= \sum_{n=0}^{\infty} \int \cdots \int_{2\pi \geq u_1 \geq u_2 \geq \cdots \geq u_n \geq 0} K(u_1) K(u_2) \cdots K(u_n) du_1 du_2 \cdots du_n. \end{aligned}$$

The above integrals converge in 'quasi-classical domain' $-\infty < C_{\text{CFT}} < -2$. For the value of C_{CFT} outside the quasi-classical domain, the integrals should be understood as analytic continuation. Let us set $U_q(\widehat{sl}_N)$ the extension of $U'_q(\widehat{sl}_N)$ by the degree operator d . Let us set $U_q(\widehat{\mathfrak{n}}^\pm)$ the extension of $U'_q(\widehat{\mathfrak{n}}^\pm)$ by the degree operator d . There exists the unique universal R -matrix $\mathcal{R} \in U_q(\widehat{\mathfrak{n}}^+) \otimes U_q(\widehat{\mathfrak{n}}^-)$ satisfying the Yang-Baxter equation,

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}.$$

The universal R 's Cartan elements \mathfrak{t} is factored as

$$\mathcal{R} = q^{\mathfrak{t}} \overline{\mathcal{R}}, \quad \mathfrak{t} = \sum_{j=1}^{N-1} h_j \otimes h^j + c \otimes d + d \otimes c,$$

where $(h^i | h_j) = \delta_{i,j}$. We call the element $\overline{\mathcal{R}} \in U'_q(\widehat{\mathfrak{n}}^+) \otimes U'_q(\widehat{\mathfrak{n}}^-)$ the reduced universal R -matrix. The \mathcal{L} -operator is an image of the reduced R -matrix [34],

$$\mathcal{L} = (sc \otimes id)(\overline{\mathcal{R}}).$$

The \mathcal{L} -operator will play an important role in trace construction of the T - Q operators.

3.2. T -operator

Let us set the T -operator $\mathbf{T}_\lambda(t)$ and $\overline{\mathbf{T}}_\lambda(t)$ by

$$\mathbf{T}_\lambda(t) = \text{Tr}_{\pi_r^{(\lambda)}} \left(\exp\left(-\pi i \sqrt{\frac{r^*}{r}} \sum_{j=1}^N P_{\omega_j} \otimes h_j\right) \mathcal{L} \right), \quad (3.2)$$

$$\overline{\mathbf{T}}_\lambda(t) = \text{Tr}_{\overline{\pi}_r^{(\lambda)}} \left(\exp\left(-\pi i \sqrt{\frac{r^*}{r}} \sum_{j=1}^N P_{\omega_j} \otimes h_j\right) \mathcal{L} \right). \quad (3.3)$$

Let us set an image of \mathcal{L} as $\mathbf{L}_\lambda(t) = (\text{id} \otimes \pi_t^{(\lambda)})(\mathcal{L})$, and the R -matrix $R_{\lambda_1, \lambda_2}(t_1/t_2) = \pi_{t_1}^{(\lambda_1)} \otimes \pi_{t_2}^{(\lambda_2)}(\mathcal{R})$. We have the so-called RLL relation,

$$R_{\lambda_1, \lambda_2}(t_1/t_2) \mathbf{L}_{\lambda_1}(t_1) \mathbf{L}_{\lambda_2}(t_2) = \mathbf{L}_{\lambda_2}(t_2) \mathbf{L}_{\lambda_1}(t_1) R_{\lambda_1, \lambda_2}(t_1/t_2).$$

Multiplying the R -matrix $R_{\lambda_1, \lambda_2}(t_1/t_2)^{-1}$ from the right, and taking trace, we have the commutation relation

$$[\mathbf{T}_{\lambda_1}(t_1), \mathbf{T}_{\lambda_2}(t_2)] = [\bar{\mathbf{T}}_{\lambda_1}(t_1), \bar{\mathbf{T}}_{\lambda_2}(t_2)] = [\mathbf{T}_{\lambda_1}(t_1), \bar{\mathbf{T}}_{\lambda_2}(t_2)] = 0.$$

The coefficients of the Taylor expansion of $\mathbf{T}_\lambda(t)$ commute with each other. Hence we have infinitely many commutative operators, which give the quantum deformation of the conservation laws of the N th KdV equation.

3.3. Q -operator

Let us set the Fock representation $\pi_j^\pm: \text{Osc}_j \rightarrow W^\pm$ with $j = 1, 2, \dots, N - 1$,

$$W^+ = \bigoplus_{k \geq 0} \mathbb{C}|k\rangle_+, \quad W^- = \bigoplus_{k \geq 0} \mathbb{C}|k\rangle_-.$$

The action is given by

$$\begin{aligned} \pi_j^+(\mathcal{H}_j)|k\rangle_+ &= -k|k\rangle_+, & \pi_j^+(\mathcal{E}_j)|k\rangle_+ &= \frac{1 - q^{-2k}}{(q - q^{-1})^2} |k - 1\rangle_+, & \pi_j^+(\mathcal{E}_j^*)|k\rangle_+ &= |k + 1\rangle_+, \\ \pi_j^-(\mathcal{H}_j)|k\rangle_- &= k|k\rangle_-, & \pi_j^-(\mathcal{E}_j)|k\rangle_- &= \frac{1 - q^{2k}}{(q - q^{-1})^2} |k - 1\rangle_-, & \pi_j^-(\mathcal{E}_j)|k\rangle_- &= |k + 1\rangle_-. \end{aligned}$$

Let π_j and $\bar{\pi}_j$ be any representation of the q -oscillator $\text{Osc} = \text{Osc}_1 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} \text{Osc}_{N-1}$ such that the partition $Z_j(t), \bar{Z}_j(t)$ converge:

$$\begin{aligned} Z_j(t) &= \text{Tr}_{\pi_j o_{t,j}} \left(\exp \left(-2\pi i \sqrt{\frac{r^*}{r}} \sum_{j=1}^N P_{\omega_j} \otimes h_j \right) \right), \\ \bar{Z}_j(t) &= \text{Tr}_{\bar{\pi}_j \bar{o}_{t,j}} \left(\exp \left(-2\pi i \sqrt{\frac{r^*}{r}} \sum_{j=1}^N P_{\omega_j} \otimes h_j \right) \right). \end{aligned}$$

Let us set the operators $\mathbf{A}_j(t)$ and $\bar{\mathbf{A}}_j(t)$ with $j = 1, 2, \dots, N$

$$\mathbf{A}_j(t) = \frac{1}{Z_j(t)} \text{Tr}_{\pi_j o_{t,j}} \left(\exp \left(-\pi i \sqrt{\frac{r^*}{r}} \sum_{j=1}^N P_{\omega_j} \otimes h_j \right) \mathcal{L} \right), \tag{3.4}$$

$$\bar{\mathbf{A}}_j(t) = \frac{1}{\bar{Z}_j(t)} \text{Tr}_{\bar{\pi}_j \bar{o}_{t,j}} \left(\exp \left(-\pi i \sqrt{\frac{r^*}{r}} \sum_{j=1}^N P_{\omega_j} \otimes h_j \right) \mathcal{L} \right). \tag{3.5}$$

Let us set Baxter's Q -operator $\mathbf{Q}_j(t)$ and $\bar{\mathbf{Q}}_j(t)$ with $j = 1, 2, \dots, N$,

$$\mathbf{Q}_j(t) = t^{-\frac{1}{2}} \sqrt{\frac{r}{r^*}}^{P_{\epsilon_j}} \mathbf{A}_j(t), \quad \bar{\mathbf{Q}}_j(t) = t^{\frac{1}{2}} \sqrt{\frac{r}{r^*}}^{P_{\epsilon_j}} \bar{\mathbf{A}}_j(t). \tag{3.6}$$

We would like to note the convenient relation

$$\begin{aligned} \sum_{k=1}^N P_{\omega_k} \otimes o_{t,j}(h_k) &= \sum_{k=1}^{N-1} (P_{\epsilon_j} - P_{\epsilon_{j+k}}) \otimes \mathcal{H}_k, \\ \sum_{k=1}^N P_{\omega_k} \otimes \bar{o}_{t,j}(h_k) &= \sum_{k=1}^{N-1} (P_{\epsilon_{j-k}} - P_{\epsilon_j}) \otimes \mathcal{H}_k. \end{aligned}$$

Here we should understand the suffix number as modulus N , i.e. $\bar{\epsilon}_{j+N} = \bar{\epsilon}_j$.

From the Yang–Baxter equation, we have the commutation relations

$$[\mathbf{Q}_{j_1}(t_1), \mathbf{Q}_{j_2}(t_2)] = [\bar{\mathbf{Q}}_{j_1}(t_1), \bar{\mathbf{Q}}_{j_2}(t_2)] = [\mathbf{Q}_{j_1}(t_1), \bar{\mathbf{Q}}_{j_2}(t_2)] = 0,$$

and

$$[\mathbf{Q}_j(t_1), \mathbf{T}_\lambda(t_2)] = [\mathbf{Q}_j(t_1), \bar{\mathbf{T}}_\lambda(t_2)] = [\bar{\mathbf{Q}}_j(t_1), \mathbf{T}_\lambda(t_2)] = [\bar{\mathbf{Q}}_j(t_1), \bar{\mathbf{T}}_\lambda(t_2)] = 0.$$

The operators $\mathbf{A}_j(t)$ can be written as power series:

$$\begin{aligned} \mathbf{A}_j(t) &= 1 + \sum_{n=1}^{\infty} \sum_{\sigma_1, \dots, \sigma_{Nn} \in \mathbb{Z}_N} a_{Nn}^{(j)}(\sigma_1, \dots, \sigma_{Nn}) \\ &\quad \times \int \cdots \int_{2\pi \geq u_1 \geq u_2 \geq \dots \geq u_{Nn} \geq 0} V_{\alpha_{\sigma_1}}(u_1) \cdots V_{\alpha_{\sigma_{Nn}}}(u_{Nn}) du_1 \cdots du_{Nn}. \end{aligned}$$

Here we have set

$$a_{Nn}^{(j)}(\sigma_1, \dots, \sigma_{Nn}) = \frac{1}{Z_j(t)} \text{Tr}_{\pi_j \circ r_j} \left(\exp \left(-2\pi i \sqrt{\frac{r^*}{r}} \sum_{j=1}^N P_{\omega_j} \otimes h_j \right) e_{\sigma_1} e_{\sigma_2} \cdots e_{\sigma_{Nn}} \right).$$

The coefficients $a_{Nn}^{(j)}$ vanish unless $n = |\{j|\sigma_j = s\}|$ for $s \in \mathbb{Z}_N$, and behave like $a_{Nn}^{(j)} \sim O(t^n)$. The coefficients $a_{Nn}^{(j)}$ are determined by the commutation relations of the Borel subalgebra $U_q(\widehat{\mathfrak{h}}^-)$ and the cyclic property of the trace, hence the specific choice of representation $\pi_j, \bar{\pi}_j$ is not significant as long as it converges. In [12, 13], we have constructed the elliptic version of the integral of the currents,

$$\int \cdots \int_{2\pi \geq u_1 \geq u_2 \geq \dots \geq u_{Nn} \geq 0} V_{\alpha_{\sigma_1}}(u_1) V_{\alpha_{\sigma_2}}(u_2) \cdots V_{\alpha_{\sigma_{Nn}}}(u_{Nn}) du_1 du_2 \cdots du_{Nn}.$$

4. Functional relations

In the previous section, we show that the T – Q operators commute with each other. In this section, we give conjecturous functional relations of the T – Q operators, which coincide with the previous work [34] upon $N = 3$ specialization. We have checked those functional relations up to the order $O(t^2)$ in the appendix. Some similar formulae have been obtained in the context of the solvable lattice models associated with $U_q(\widehat{sl}_N)$ [38–40]. At the end of this section we summarize the conclusion.

4.1. Functional relations

The T -operator is written by the determinant of the Q -operators. Let us set the Young diagram $\mu = (\mu_1, \mu_2, \dots, \mu_N)$, ($\mu_j \geq \mu_{j+1}; \mu_j \in \mathbb{N}$). Using the same character as the Young diagram μ , we represent the highest weight $\mu = \mu_1 \Lambda_1 + \cdots + \mu_N \Lambda_N$. We set $c_0 = \prod_{1 \leq j < k \leq N} \left(\sqrt{\frac{z_j}{z_k}} - \sqrt{\frac{z_k}{z_j}} \right)$. We have the following determinant formulae of the T -operator:

$$\mathbf{T}_\mu(t) = \frac{1}{c_0} \begin{vmatrix} \mathbf{Q}_1(tq^{2\tilde{\mu}_1}) & \mathbf{Q}_1(tq^{2\tilde{\mu}_2}) & \cdots & \mathbf{Q}_1(tq^{2\tilde{\mu}_N}) \\ \mathbf{Q}_2(tq^{2\tilde{\mu}_1}) & \mathbf{Q}_2(tq^{2\tilde{\mu}_2}) & \cdots & \mathbf{Q}_2(tq^{2\tilde{\mu}_N}) \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{Q}_N(tq^{2\tilde{\mu}_1}) & \mathbf{Q}_N(tq^{2\tilde{\mu}_2}) & \cdots & \mathbf{Q}_N(tq^{2\tilde{\mu}_N}) \end{vmatrix}, \quad (4.1)$$

$$\bar{\mathbf{T}}_\mu(t) = \frac{1}{c_0} \begin{vmatrix} \bar{\mathbf{Q}}_1(tq^{-2\tilde{\mu}_1}) & \bar{\mathbf{Q}}_1(tq^{-2\tilde{\mu}_2}) & \dots & \bar{\mathbf{Q}}_1(tq^{-2\tilde{\mu}_N}) \\ \bar{\mathbf{Q}}_2(tq^{-2\tilde{\mu}_1}) & \bar{\mathbf{Q}}_2(tq^{-2\tilde{\mu}_2}) & \dots & \bar{\mathbf{Q}}_2(tq^{-2\tilde{\mu}_N}) \\ \dots & \dots & \dots & \dots \\ \bar{\mathbf{Q}}_N(tq^{-2\tilde{\mu}_1}) & \bar{\mathbf{Q}}_N(tq^{-2\tilde{\mu}_2}) & \dots & \bar{\mathbf{Q}}_N(tq^{-2\tilde{\mu}_N}) \end{vmatrix}. \tag{4.2}$$

Here we have used the auxiliary parameters $2\tilde{\mu}_j = 2\mu_j + N - 2j + 1, (1 \leq j \leq N)$. We have checked the above formulae (4.1) and (4.2) for $\mu = \Lambda_1$ and $\mu = \Lambda_1 + \dots + \Lambda_{N-1}$, up to the order $O(t^2)$ (see the appendix). As the special case $\mu_j = 0, (1 \leq j \leq N)$, we have the quantum Wronskian condition:

$$c_0 = \begin{vmatrix} \mathbf{Q}_1(tq^{N-1}) & \mathbf{Q}_1(tq^{N-3}) & \dots & \mathbf{Q}_1(tq^{-N+1}) \\ \mathbf{Q}_2(tq^{N-1}) & \mathbf{Q}_2(tq^{N-3}) & \dots & \mathbf{Q}_2(tq^{-N+1}) \\ \dots & \dots & \dots & \dots \\ \mathbf{Q}_N(tq^{N-1}) & \mathbf{Q}_N(tq^{N-3}) & \dots & \mathbf{Q}_N(tq^{-N+1}) \end{vmatrix}, \tag{4.3}$$

$$c_0 = \begin{vmatrix} \bar{\mathbf{Q}}_1(tq^{-N+1}) & \bar{\mathbf{Q}}_1(tq^{-N+3}) & \dots & \bar{\mathbf{Q}}_1(tq^{N-1}) \\ \bar{\mathbf{Q}}_2(tq^{-N+1}) & \bar{\mathbf{Q}}_2(tq^{-N+3}) & \dots & \bar{\mathbf{Q}}_2(tq^{N-1}) \\ \dots & \dots & \dots & \dots \\ \bar{\mathbf{Q}}_N(tq^{-N+1}) & \bar{\mathbf{Q}}_N(tq^{-N+3}) & \dots & \bar{\mathbf{Q}}_N(tq^{N-1}) \end{vmatrix}. \tag{4.4}$$

We have checked the above formulae (4.3) and (4.4), up to the order $O(t^2)$ (see the appendix). Let us set $c_i = \prod_{1 \leq j < k \leq N, j, k \neq i} (\sqrt{\frac{z_j}{z_k}} - \sqrt{\frac{z_k}{z_j}})$ for $1 \leq i \leq N$. The two kinds of Q -operator, $\mathbf{Q}_j(t)$ and $\bar{\mathbf{Q}}_j(t)$, are functionally dependent. The Q -operator $\mathbf{Q}_i(t)$ is written by the determinant of the Q -operator $\bar{\mathbf{Q}}_j(t)$,

$$c_i \mathbf{Q}_i(t) = \begin{vmatrix} \bar{\mathbf{Q}}_1(tq^{N-2}) & \bar{\mathbf{Q}}_1(tq^{N-4}) & \dots & \bar{\mathbf{Q}}_1(tq^{-N+2}) \\ \dots & \dots & \dots & \dots \\ \bar{\mathbf{Q}}_{i-1}(tq^{N-2}) & \bar{\mathbf{Q}}_{i-1}(tq^{N-4}) & \dots & \bar{\mathbf{Q}}_{i-1}(tq^{-N+2}) \\ \bar{\mathbf{Q}}_{i+1}(tq^{N-2}) & \bar{\mathbf{Q}}_{i+1}(tq^{N-4}) & \dots & \bar{\mathbf{Q}}_{i+1}(tq^{-N+2}) \\ \dots & \dots & \dots & \dots \\ \bar{\mathbf{Q}}_N(tq^{N-2}) & \bar{\mathbf{Q}}_N(tq^{N-4}) & \dots & \bar{\mathbf{Q}}_N(tq^{-N+2}) \end{vmatrix}, \tag{4.5}$$

$$c_i \bar{\mathbf{Q}}_i(t) = \begin{vmatrix} \mathbf{Q}_1(tq^{-N+2}) & \mathbf{Q}_1(tq^{-N+4}) & \dots & \mathbf{Q}_1(tq^{N-2}) \\ \dots & \dots & \dots & \dots \\ \mathbf{Q}_{i-1}(tq^{-N+2}) & \mathbf{Q}_{i-1}(tq^{-N+4}) & \dots & \mathbf{Q}_{i-1}(tq^{N-2}) \\ \mathbf{Q}_{i+1}(tq^{-N+2}) & \mathbf{Q}_{i+1}(tq^{-N+4}) & \dots & \mathbf{Q}_{i+1}(tq^{N-2}) \\ \dots & \dots & \dots & \dots \\ \mathbf{Q}_N(tq^{-N+2}) & \mathbf{Q}_N(tq^{-N+4}) & \dots & \mathbf{Q}_N(tq^{N-2}) \end{vmatrix}, \tag{4.6}$$

with $i = 1, 2, \dots, N$. We have checked the determinant formulae (4.5) and (4.6) up to the order $O(t^2)$ (see the appendix). We derive the following (4.7)–(4.13) from the above formulae (4.1), (4.2), (4.5) and (4.6). We have the higher-rank generalization of Baxter’s T – Q relation (4.7) and (4.8), as the consequence of (4.1) and (4.2),

$$\mathbf{Q}_i(tq^N) + \sum_{s=1}^{N-1} (-1)^s \mathbf{T}_{\Lambda_1 + \dots + \Lambda_s}(tq^{-1}) \mathbf{Q}_i(tq^{N-2s}) + (-1)^N \mathbf{Q}_i(tq^{-N}) = 0, \tag{4.7}$$

$$\bar{\mathbf{Q}}_i(tq^{-N}) + \sum_{s=1}^{N-1} (-1)^s \bar{\mathbf{T}}_{\Lambda_1+\dots+\Lambda_s}(tq) \bar{\mathbf{Q}}_i(tq^{-N+2s}) + (-1)^N \bar{\mathbf{Q}}_i(tq^N) = 0, \tag{4.8}$$

with $i = 1, 2, \dots, N$. This Baxter’s T – Q relation, (4.7) and (4.8), coincides with those in [34] upon $N = 3$ specialization. Note that the specialization to $N = 2$ does not yield the formulae in [28–30], because the Dynkin diagram for $N = 2$ is different from those for $N \geq 3$. We have to give separate definitions of the bosons, the q -oscillator and the screening currents for $N = 2$, [28–30]. This Baxter’s T – Q relation (4.7), (4.8) coincides with those of [38] for $N \geq 3$. In [38], Krichever *et al* gave the conjecture that the standard objects of quantum integrable models are identified with elements of classical nonlinear integrable difference equation. For the simplest example they showed that the fusion rules for quantum transfer matrices coincide with the Hirota–Miwa’s bilinear difference equation [45, 46] (the discrete KP). They derived higher-rank generalization of Baxter’s T – Q relation by analysing the Hirota–Miwa’s bilinear difference equation (classical nonlinear integrable difference equation) too. In this paper, we derive the same Baxter’s T – Q relation by analysing the quantum field theory of the KP (quantum integrable model). Hence this paper gives a supporting argument of the conjecture on quantum and classical discrete integrable models, by Krichever *et al* [38]. As a consequence of (4.5) and (4.6), we have the bilinear formulae of the T -operator (4.9) and (4.10):

$$(-1)^{\frac{(N-1)(N-2)}{2}} c_0 \mathbf{T}_{m\Lambda_1}(t) = \sum_{s=1}^N (-1)^{s+1} c_s \mathbf{Q}_s(tq^{2m+N-1}) \bar{\mathbf{Q}}_s(tq^{-1}), \tag{4.9}$$

$$(-1)^{\frac{(N-1)(N-2)}{2}} c_0 \bar{\mathbf{T}}_{m\Lambda_1}(t) = \sum_{s=1}^N (-1)^{s+1} c_s \bar{\mathbf{Q}}_s(tq^{-2m-N+1}) \mathbf{Q}_s(tq), \tag{4.10}$$

and

$$(-1)^{\frac{(N-1)(N-2)}{2}} c_0 \mathbf{T}_{m(\Lambda_1+\dots+\Lambda_{N-1})}(t) = \sum_{s=1}^N (-1)^{N+s} c_s \bar{\mathbf{Q}}_s(tq^{-2m-1}) \bar{\mathbf{Q}}_s(tq^{N-1}), \tag{4.11}$$

$$(-1)^{\frac{(N-1)(N-2)}{2}} c_0 \bar{\mathbf{T}}_{m(\Lambda_1+\dots+\Lambda_{N-1})}(t) = \sum_{s=1}^N (-1)^{N+s} c_s \mathbf{Q}_s(tq^{2m+1}) \bar{\mathbf{Q}}_s(tq^{-N+1}). \tag{4.12}$$

As a consequence of the determinant formulae (4.1) and (4.2), we have the Jacobi–Trudi formulae of the T -operator. For the Young diagram $\mu = (\mu_1, \mu_1, \dots, \mu_{N-1}, 0)$, we have

$$\mathbf{T}_\mu(t) = \begin{vmatrix} \tau^{(\mu'_1)}(t) & \dots & \tau^{(\mu'_1+j-1)}(tq^{2(j-1)}) & \dots & \tau^{(\mu'_1+l(\mu')-1)}(tq^{2(l(\mu')-1)}) \\ \dots & \dots & \dots & \dots & \dots \\ \tau^{(\mu'_i-i+1)}(t) & \dots & \tau^{(\mu'_i-i+j)}(tq^{2(j-1)}) & \dots & \tau^{(\mu'_i-i+l(\mu'))}(tq^{2(l(\mu')-1)}) \\ \dots & \dots & \dots & \dots & \dots \\ \tau^{(\mu'_{l(\mu')-l(\mu')+1})}(t) & \dots & \tau^{(\mu'_{l(\mu')-l(\mu')+j})}(tq^{2(j-1)}) & \dots & \tau^{(\mu'_{l(\mu')})}(tq^{2(l(\mu')-1)}) \end{vmatrix}. \tag{4.13}$$

Here we have set $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_N)$ the transpose Young diagram of μ , and $l(\mu') = \mu_1$. We have set $\tau^{(s)}(t) = \mathbf{T}_{\Lambda_1+\dots+\Lambda_s}(t)$. We have $\tau^{(0)}(t) = \tau^{(N)}(t) = 1$. The above conjecturous functional relations of the T – Q operators, (4.1)–(4.13), coincide with the previous work [34] upon $N = 3$ specialization.

4.2. Conclusion

In this paper we present a q -oscillator representation of the Borel subalgebra $U'_q(\widehat{sl}_N)$, (2.2). By using this q -oscillator representation, we give the free field realization of Baxter's Q -operator $\mathbf{Q}_j(t), \overline{\mathbf{Q}}_j(t)$ with $j = 1, 2, \dots, N$, for the W_N -algebra, (3.4)–(3.6). The commutativity of the Q -operator is direct consequence of the Yang–Baxter equation. We give conjecturous determinant formulae of the T – Q operators for the W_N -algebra, (4.1), (4.2), (4.5), (4.6), which produce the higher-rank W_N -generalization of Baxter's T – Q relation, (4.7), (4.8). We have checked these determinant formulae of the T – Q operators, (4.1), (4.2), (4.5), (4.6) up to the order $O(t^2)$ in the appendix. Because the scheme of functional relations works well, we conclude that the number of the Q -operators for the W_N -algebra is just $2N$, ($N \geq 3$). In this paper we did not give a complete proof of the determinant formulae for the W_N -algebra. Bazhanov *et al* [34] gave the proof of the determinant formulae for the W_3 -algebra. Their proof is based on the trace of the universal \mathcal{L} -operator over Verma module, and the Bernstein–Gel'fand–Gel'fand (BGG) resolution. Because we have already established conjecturous determinant formulae, higher-rank generalization of the complete proof seems a calculation problem. However it is not so easy.

Acknowledgments

The author would like to thank Professors V Bazhanov and M Jimbo for useful communications. The author would like to thank the Institute of Advanced Studies, Australian National University for the hospitality during his visit to Canberra in March 2008. The author would like to thank Professors P Bouwknegt, A Chervov, V Gerdjikov, K Hasegawa, W-X Ma and V Mangazeev for their interests in this work. This work is partly supported by Grant-in Aid for Young Scientist B (18740092) from JSPS.

Appendix. Supporting arguments

In this appendix, we give some supporting arguments on conjecturous formulae of the determinant formulae (4.1)–(4.6). We check those determinant formulae up to the order $O(t^2)$. At first we prepare the Taylor expansion of $\mathbf{A}_j(t), \overline{\mathbf{A}}_j(t)$. Let us set $\pi_j = \overline{\pi}_j = \pi_1^+ \otimes \dots \otimes \pi_{N-1}^+$. Taking the trace for the basis $\{|n_1, n_2, \dots, n_{N-1}\rangle = (\mathcal{H}_1^*)^{n_1} (\mathcal{H}_2^*)^{n_2} \dots (\mathcal{H}_{N-1}^*)^{n_{N-1}} |0\rangle_+ \otimes \dots \otimes |0\rangle_+\}_{n_1, n_2, \dots, n_{N-1} \in \mathbb{N}}$, we have

$$Z_j(t) = \text{Tr}_{\pi_j} \left(\exp \left(-2\pi i \sqrt{\frac{r^*}{r}} \sum_{k=1}^{N-1} (P_{\bar{\epsilon}_j} - P_{\bar{\epsilon}_{j+k}}) \otimes \mathcal{H}_k \right) \right) = \prod_{\substack{k=1 \\ k \neq j}}^N \left(1 - \frac{z_k}{z_j} \right)^{-1},$$

with $j = 1, 2, \dots, N$. In the same manner as the above, we have

$$\overline{Z}_j(t) = \prod_{\substack{k=1 \\ k \neq j}}^N \left(1 - \frac{z_j}{z_k} \right)^{-1}.$$

Let us set a_i, \bar{a}_i by

$$\mathbf{A}_i(t) = 1 + a_i t + O(t^2), \quad \overline{\mathbf{A}}_i(t) = 1 + \bar{a}_i t + O(t^2).$$

Let us set

$$J_{k_1, k_2, \dots, k_N}^{(n)} = \int \cdots \int_{2\pi \geq u_1 \geq u_2 \geq \dots \geq u_{Nn} \geq 0} V_{k_1}(u_1) V_{k_2}(u_2) \cdots V_{k_N}(u_N) \\ \times V_{k_1}(u_{N+1}) V_{k_2}(u_{N+2}) \cdots V_{k_N}(u_{2N}) \cdots \\ \times V_{k_1}(u_{N(n-1)+1}) V_{k_2}(u_{N(n-1)+2}) \cdots V_{k_N}(u_{Nn}) du_1 du_2 \cdots du_{Nn}.$$

Let us calculate coefficient of $\mathcal{J}_{1,2,\dots,N}^{(1)}$ in a_i . We have

$$\text{Tr}_{\pi_i} \left(o_{i,i} \left(\exp \left(-2\pi i \sqrt{\frac{r^*}{r}} \sum_{k=1}^{N-1} P_{\omega_k} \otimes h_k \right) e_1 e_2 e_3 \cdots e_N \right) \right) \times \mathcal{J}_{1,2,\dots,N}^{(1)} = t(q - q^{-1})^{N-2} q^{\frac{N-2}{2}} \\ \times \prod_{k=1}^{N-1} \text{Tr}_{\pi_k^+} \left(\exp \left(-2\pi i \sqrt{\frac{r^*}{r}} (P_{\bar{\epsilon}_i} - P_{\bar{\epsilon}_{i+k}}) \otimes \mathcal{H}_k \right) \mathcal{E}_k \mathcal{E}_k^* \right) \times \mathcal{J}_{1,2,\dots,N}^{(1)}.$$

Taking the trace and dividing $Z_i(t)$, we have

$$a_i = \frac{q^{\frac{3}{2}N-2} z_i^{N-2} z_1}{(q - q^{-1}) \prod_{\substack{k=1 \\ k \neq i}}^N (q^2 z_i - z_k)} \times \mathcal{J}_{1,2,\dots,N}^{(1)} + \cdots,$$

with $i = 1, 2, \dots, N$. In the same manner as the above, we have

$$\bar{a}_i = (-1)^N \frac{q^{\frac{1}{2}N} z_i^{N-2} z_1}{(q - q^{-1}) \prod_{\substack{k=1 \\ k \neq i}}^N (q^2 z_k - z_i)} \times \mathcal{J}_{1,2,\dots,N}^{(1)} + \cdots,$$

with $i = 1, 2, \dots, N$. Let us check the determinant relations between $\mathbf{Q}_i(t)$ and $\bar{\mathbf{Q}}_i(t)$, (4.5) and (4.6). We have

$$\begin{vmatrix} \mathbf{Q}_1(tq^{N-2}) & \mathbf{Q}_1(tq^{N-4}) & \cdots & \mathbf{Q}_1(tq^{-N+2}) \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{Q}_{i-1}(tq^{N-2}) & \mathbf{Q}_{i-1}(tq^{N-3}) & \cdots & \mathbf{Q}_{i-1}(tq^{-N+2}) \\ \mathbf{Q}_{i+1}(tq^{N-2}) & \mathbf{Q}_{i+1}(tq^{N-3}) & \cdots & \mathbf{Q}_{i+1}(tq^{-N+2}) \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{Q}_N(tq^{N-2}) & \mathbf{Q}_N(tq^{N-4}) & \cdots & \mathbf{Q}_N(tq^{-N+2}) \end{vmatrix} = t^{-\sqrt{\frac{r}{r^*}} P_{\bar{\epsilon}_i}} \prod_{\substack{1 \leq j < k \leq N \\ j, k \neq i}} \left(\sqrt{\frac{z_j}{z_k}} - \sqrt{\frac{z_k}{z_j}} \right) \\ \times \left(1 + \left(\sum_{\substack{j=1 \\ j \neq i}}^N a_j \prod_{\substack{k=1 \\ k \neq i, j}}^N \frac{(q^2 z_j - z_k)}{(z_j - z_k)} q^{-N+2} \right) t + O(t^2) \right).$$

Inserting the formulae of a_i into the RHS and using the following identity,

$$\sum_{\substack{j=1 \\ j \neq i}}^N \frac{z_j^{N-2}}{(z_j q^2 - z_i) \prod_{\substack{k=1 \\ k \neq i, j}}^N (z_j - z_k)} = (-1)^N \frac{z_i^{N-2}}{\prod_{\substack{k=1 \\ k \neq i}}^N (z_k q^2 - z_i)},$$

we have

$$t^{-\sqrt{\frac{r}{r^*}} P_{\bar{\epsilon}_i}} c_i \left(1 + (-1)^N \frac{q^{\frac{1}{2}N} z_i^{N-2} z_1}{(q - q^{-1}) \prod_{\substack{k=1 \\ k \neq i}}^N (z_k q^2 - z_i)} \times \mathcal{J}_{1,2,\dots,N}^{(1)} \times t + \cdots \right),$$

which coincides with the leading terms of $\bar{\mathbf{Q}}_i(t)$. As the same argument as the above, the coefficients of $\mathcal{J}_{k_1, k_2, \dots, k_N}^{(1)}$ coincide with each other up to the order $O(t^2)$. Now we have

checked the determinant formulae (4.5) and (4.6) up to the order $O(t^2)$. For the second we check the quantum Wronskian condition (4.3) and (4.4) up to the order $O(t^2)$. We have Taylor expansion of determinant of $\mathbf{Q}_j(t)$,

$$\begin{aligned} & \begin{vmatrix} \mathbf{Q}_1(tq^{N-1}) & \mathbf{Q}_1(tq^{N-3}) & \cdots & \mathbf{Q}_1(tq^{-N+1}) \\ \mathbf{Q}_2(tq^{N-1}) & \mathbf{Q}_2(tq^{N-3}) & \cdots & \mathbf{Q}_2(tq^{-N+1}) \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{Q}_N(tq^{N-1}) & \mathbf{Q}_N(tq^{N-3}) & \cdots & \mathbf{Q}_N(tq^{-N+1}) \end{vmatrix} \\ &= \prod_{1 \leq j < k \leq N} \left(\sqrt{\frac{z_j}{z_k}} - \sqrt{\frac{z_k}{z_j}} \right) \left(1 + \left(\sum_{i=1}^N a_i \prod_{\substack{k=1 \\ k \neq i}}^N \frac{(q^2 z_i - z_k)}{(z_i - z_k)} q^{-N+1} \right) t + O(t^2) \right). \end{aligned}$$

Inserting the explicit formulae of a_i into RHS and using the following identity,

$$\sum_{i=1}^N (-1)^{i+1} z_i^{N-2} \prod_{\substack{1 \leq j < k \leq N \\ j, k \neq i}} (z_j - z_k) = 0,$$

we have

$$\prod_{1 \leq j < k \leq N} \left(\sqrt{\frac{z_j}{z_k}} - \sqrt{\frac{z_k}{z_j}} \right) (1 + O(t^2)).$$

Now we have checked the quantum Wronskian condition (4.3) and (4.4) up to the order $O(t^2)$. Next we consider the determinant formulae (4.1) and (4.2) for the special cases $\mu = \Lambda_1$ and $\mu = \Lambda_1 + \cdots + \Lambda_{N-1}$. Because we have checked formulae (4.5) and (4.6) up to the order $O(t^2)$, it is enough to show (4.9)–(4.12) in order to show (4.1) and (4.2) up to the order $O(t^2)$. We have

$$\begin{aligned} & (-1)^{\frac{(N-1)(N-2)}{2}} \sum_{s=1}^N (-1)^{s+1} c_s \mathbf{Q}_s(tq^{N+1}) \overline{\mathbf{Q}}_s(tq^{-1}) \\ &= \sum_{s=1}^N (-1)^{s+1} z_s^N \prod_{\substack{1 \leq j < k \leq N \\ j, k \neq s}} (z_j - z_k) (1 + t(q^{N+1} a_s + q^{-1} \bar{a}_s) + O(t^2)). \end{aligned}$$

Using the following relation,

$$\sum_{s=1}^N (-1)^{s+1} z_s^N \prod_{\substack{1 \leq j < k \leq N \\ j, k \neq s}} (z_j - z_k) = \prod_{1 \leq j < k \leq N} (z_j - z_k) (z_1 + z_2 + \cdots + z_N),$$

we show that the first leading term becomes $c_0 \sum_{j=1}^N z_j$. Inserting the explicit formulae of a_s, \bar{a}_s and using the following relation,

$$\sum_{s=1}^N \frac{z_s^{2N-2}}{\prod_{\substack{k=1 \\ k \neq s}}^N (z_s - z_k)} \left(\frac{q}{\prod_{\substack{k=1 \\ k \neq s}}^N (z_s - q^2 z_k)} - \frac{q^{-1}}{\prod_{\substack{k=1 \\ k \neq s}}^N (z_s - q^{-2} z_k)} \right) = (q - q^{-1}),$$

we have the second leading term,

$$tq^{\frac{Nq}{2}} \prod_{1 \leq j < k \leq N} (z_k - z_j) (z_1 \mathcal{J}_{1,2,\dots,N}^{(1)} + z_2 \mathcal{J}_{2,3,\dots,N,1}^{(1)} + \cdots).$$

Now we need explicit formulae of $\mathbf{T}_{\Lambda_1}(t)$. Let us fix a basis of the irreducible highest representation of $U_q(\mathfrak{gl}_N)$ with Λ_1 by

$$|\Lambda_1\rangle, \quad \pi^{(\Lambda_1)}(E_{\alpha_1})|\Lambda_1\rangle, \quad \pi^{(\Lambda_1)}(E_{\alpha_2}E_{\alpha_1})|\Lambda_1\rangle, \dots, \pi^{(\Lambda_1)}(E_{\alpha_{N-1}} \dots E_{\alpha_1})|\Lambda_1\rangle.$$

The matrix representation of $\pi^{(\Lambda_1)}$ is written upon this basis by

$$\begin{aligned} \pi^{(\Lambda_1)}(E_{\alpha_i}) &= (\delta_{j,i}\delta_{k,i+1})_{1 \leq j,k \leq N}, & (1 \leq i \leq N-1), \\ \pi^{(\Lambda_1)}(F_{\alpha_i}) &= (\delta_{j,i+1}\delta_{k,i})_{1 \leq j,k \leq N}, & (1 \leq i \leq N-1), \\ \pi^{(\Lambda_1)}(H_i) &= (\delta_{j,i}\delta_{k,i})_{1 \leq j,k \leq N}, & (1 \leq i \leq N). \end{aligned}$$

Using this matrix representation, we have

$$\mathbf{T}_{\Lambda_1}(t) = \sum_{j=1}^N z_j + \sum_{n=1}^{\infty} t^n q^{\frac{Nn}{2}} \sum_{j=1}^N z_j J_{j,j+1,j+2,\dots,j+N-1}^{(n)}.$$

Now we have checked the determinant formula (4.1) for $\mu = \Lambda_1$ up to the order $O(t^2)$. As the same manner we checked the determinant formula (4.1) for $\mu = \Lambda_1 + \dots + \Lambda_{N-1}$ and (4.2) for $\mu = \Lambda_1, \Lambda_1 + \dots + \Lambda_{N-1}$, up to the order $O(t^2)$. For reader's convenience we summarize the explicit formulae of $\mathbf{T}_{\Lambda_1 + \dots + \Lambda_{N-1}}(t)$, $\bar{\mathbf{T}}_{\Lambda_1}(t)$ and $\bar{\mathbf{T}}_{\Lambda_1 + \dots + \Lambda_{N-1}}(t)$. The matrix representation of $\pi^{(\Lambda_1 + \dots + \Lambda_{N-1})}$ is written by

$$\begin{aligned} \pi^{(\Lambda_1 + \dots + \Lambda_{N-1})}(E_{\alpha_i}) &= (\delta_{j,N-i}\delta_{k,N-i+1})_{1 \leq j,k \leq N}, & (1 \leq i \leq N-1), \\ \pi^{(\Lambda_1 + \dots + \Lambda_{N-1})}(F_{\alpha_i}) &= (\delta_{j,N-i+1}\delta_{k,N-i})_{1 \leq j,k \leq N}, & (1 \leq i \leq N-1), \\ \pi^{(\Lambda_1 + \dots + \Lambda_{N-1})}(H_i - H_{i+1}) &= (\delta_{j,N-i}\delta_{k,N-i} - \delta_{j,N-i+1}\delta_{k,N-i+1})_{1 \leq j,k \leq N}, & (1 \leq i \leq N-1). \end{aligned}$$

We have

$$\begin{aligned} \mathbf{T}_{\Lambda_1 + \dots + \Lambda_{N-1}}(t) &= \sum_{j=1}^N \frac{1}{z_j} + \sum_{n=1}^{\infty} t^n q^{(2-\frac{N}{2})n} (-1)^{Nn} \sum_{j=1}^N \frac{1}{z_j} J_{j+N-1,\dots,j+1,j}^{(n)}, \\ \bar{\mathbf{T}}_{\Lambda_1}(t) &= \sum_{j=1}^N z_j + \sum_{n=0}^{\infty} t^n q^{-\frac{Nn}{2}} \sum_{j=1}^N z_j J_{j,j+1,j+2,\dots,j+N-1}^{(n)}, \\ \bar{\mathbf{T}}_{\Lambda_1 + \dots + \Lambda_{N-1}}(t) &= \sum_{j=1}^N \frac{1}{z_j} + \sum_{n=1}^{\infty} t^n q^{(\frac{N}{2}-2)n} \sum_{j=1}^N \frac{1}{z_j} J_{j+N-1,\dots,j+1,j}^{(n)}. \end{aligned}$$

Using these explicit formulae, we have

$$\begin{aligned} \mathbf{T}_{\Lambda_1}(q^{-\frac{N}{2}}t) &= \bar{\mathbf{T}}_{\Lambda_1}(q^{\frac{N}{2}}t), \\ \mathbf{T}_{\Lambda_1 + \dots + \Lambda_{N-1}}(q^{\frac{N-4}{2}}t) &= \bar{\mathbf{T}}_{\Lambda_1 + \dots + \Lambda_{N-1}}(q^{\frac{4-N}{2}}t), \\ \mathbf{T}_0(t) &= \bar{\mathbf{T}}_0(t) = 1. \end{aligned}$$

References

- [1] Baxter R 1972 Partition function of the eight-vertex model *Ann. Phys.* **70** 193–228
- [2] Baxter R 1973 Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain: I. Some fundamental eigenvectors *Ann. Phys.* **76** 1–24
- [3] Baxter R 1973 Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain: II. Equivalence to a generalized ice-type lattice model *Ann. Phys.* **76** 25–47
- [4] Baxter R 1973 Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain: III. Eigenvectors of the transfer matrix and the Hamiltonian *Ann. Phys.* **76** 48–71

- [5] Baxter R 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
- [6] Takhtadzhian L and Faddeev L 1979 The quantum method of the inverse problem and the Heisenberg XYZ model *Russ. Math. Surv.* **34** 11–68
- [7] Fabricius K and McCoy B 2003 New development in the eight vertex model *J. Stat. Phys.* **111** 323–7
- [8] Fabricius K and McCoy B 2004 Functional equations and fusion matrices for the eight vertex model *Pub. Res. Inst. Math. Sci.* **40** 905–32
- [9] Fabricius K and McCoy B 2006 An elliptic current operator for the eight vertex model *J. Phys. A: Math. Gen.* **39** 14869–86
- [10] Fabricius K 2007 A new Q -matrix in the eight vertex model *J. Phys. A: Math. Theor.* **40** 4075–86
- [11] Bazhanov V and Mangazeev V 2007 Analytic theory of the eight-vertex model *Nucl. Phys. B* **775** 225–82
- [12] Feigin B, Kojima T, Shiraishi J and Watanabe H 2006 The integrals of motion for the deformed W -algebra $W_{q,t}(\widehat{sl}_N)$ *Proc. for Representation Theory (Atami, Japan)*
- [13] Kojima T and Shiraishi J 2007 The integrals of motion for the deformed W -algebra $W_{q,t}(\widehat{sl}_N)$: II. Proof of commutation relations *Commun. Math. Phys.* (Preprint [arXiv:0709.2305](https://arxiv.org/abs/0709.2305)) at press
- [14] Sklyanin E 1985 The quantum Toda chain *Lecture Notes in Physics* **226** 196–223
- [15] Sklyanin E 1990 *Functional Bethe Ansatz, Integrable and Superintegrable systems* (Teaneck, NJ: World Scientific) pp 8–33
- [16] Sklyanin E 1995 Separation of variables—new trends. Quantum field theory, integrable models and beyond (Kyoto) *Prog. Theor. Phys. Suppl.* **118** 35–60
- [17] Kuznetsov V, Mangazeev V and Sklyanin E 2003 Q -operator and factorized separation chain for Jack polynomials *Indagat. Math.* **14** 451–82
- [18] Pasquier V and Gaudin M 1992 The periodic Toda chain and matrix generalization of the Bessel function recursion relations *J. Phys. A: Math. Gen.* **25** 5243–52
- [19] Derkachov S 1999 Baxter’s Q -operator for the homogeneous XXX spin chain *J. Phys. A: Math. Gen.* **32** 5299–316
- [20] Derkachov S, Korchemsky G and Manashov A 2001 Noncompact Heisenberg spin magnets from high-energy QCD: I. Baxter Q -operator and separation of variables *Nucl. Phys. B* **617** 375–440
- [21] Derkachov S, Korchemsky G and Manashov A 2003 Baxter Q -operator and Separation of Variables for the open $SL(2, \mathbb{R})$ spin chain *J. High. Energy. Phys.* **JHEP2003(2003)31** (no 10, paper 053) (electronic)
- [22] Derkachov S, Karakhanyan D and Kirschner R 2006 Baxter Q -operators of the XXZ chain and R -matrix factorization *Nucl. Phys. B* **738** 368–90
- [23] Derkachov S and Manashov A 2006 R -matrix and Baxter Q -operators for the noncompact $sl(N, \mathbb{C})$ invariant spin chain *SIGMA Symmetry Integrability Geom. Methods Appl.* paper 084, 20 pp (electronic)
- [24] Belitsky A, Derkachov S, Korchemsky G and Manashov A 2007 The Baxter Q -operator for the graded $SL(2|1)$ spin chain *J. Stat. Mech. Theor. Exp.* 2007 paper1005, 63 pp (electronic)
- [25] Korff C 2006 A Q -Operator identity for the correlation functions of the infinite XXZ spin-chain *J. Phys. A: Math. Gen.* **39** 3203–19
- [26] Korff C 2007 A Q -operator for the quantum transfer matrix *J. Phys. A: Math. Theor.* **40** 3749–74
- [27] Bytsko A and Teschner J 2006 Quantization of models with non-compact quantum group symmetry, modular XXZ magnet and lattice sinh Gordon model *J. Phys. A: Math. Gen.* **39** 12927–81
- [28] Bazhanov V, Lukyanov S and Zamolodchikov AI 1996 Integrable structure of conformal field theory: quantum KdV theory and thermodynamic Bethe ansatz *Commun. Math. Phys.* **177** 381–98
- [29] Bazhanov V, Lukyanov S and Zamolodchikov AI 1997 Integrable structure of conformal field theory: II. Q -operator and DDV equation *Commun. Math. Phys.* **190** 247–78
- [30] Bazhanov V, Lukyanov S and Zamolodchikov AI 1999 Integrable structure of conformal field theory: III. The Yang–Baxter relation *Commun. Math. Phys.* **200** 297–324
- [31] Bazhanov V, Lukyanov S and Zamolodchikov AI 2001 Spectral determinant for Schrödinger equation and Q -operator of conformal field theory *J. Stat. Phys.* **102** 567–76
- [32] Rossi M and Weston R 2002 A generalized Q -operator for $U_q(\widehat{sl}_2)$ -vertex model *J. Phys. A: Math. Gen.* **35** 10015–32
- [33] Dorey P and Tateo R 1999 Anharmonic oscillators, the thermodynamic Bethe ansatz and nonlinear integral equation *J. Phys. A: Math. Gen.* **32** L419–L425
- [34] Bazhanov V, Hibberd A and Khoroshkin S 2002 Integrable structure of W_3 conformal field theory, quantum Boussinesq theory and boundary affine Toda theory *Nucl. Phys. B* **622** 475–547
- [35] Kulish P and Zeitlin Z 2005 Superconformal field theory and SUSY $N = 1$ KdV hierarchy: II. The Q -operator *Nucl. Phys. B* **709** 578–91
- [36] Antonov A and Feigin B 1997 Quantum group representation and Baxter equation *Phys. Lett. B* **392** 115–22

- [37] Asai Y, Jimbo M, Miwa T and Pugai Y 1996 Bosonization of vertex operators for $A_{n-1}^{(1)}$ face model *J. Phys. A: Math. Gen.* **29** 6595–616
- [38] Krichever I, Lipan O, Wiegmann P and Zabrodin A 1997 Quantum integrable models and discrete classical Hirota equations *Commun. Math. Phys.* **188** 267–304
- [39] Bazhanov V and Reshetikhin N 1990 Restricted solid-on-solid models connected with simply laced algebras and conformal field theory *J. Phys. A: Math. Gen.* **23** 1477–92
- [40] Kuniba A, Nakanishi T and Suzuki J 1994 Functional relations in solvable lattice models: I. Functional relations and representation theory *Int. J. Mod. Phys. A* **9** 5215–66
- [41] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2007 Hidden grassmann structure in the XXZ model *Commun. Math. Phys.* **272** 263–81
- [42] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2008 Hidden grassmann structure in the XXZ model: II. Creation operators *Preprint* [arXiv:0801.1176](https://arxiv.org/abs/0801.1176)
- [43] Chervov A and Falqui G 2007 Manin matrices and Talalaev’s formula *Preprint* [arXiv:0711.2236](https://arxiv.org/abs/0711.2236)
- [44] Fateev V and Lukyanov S 1988 The models of two-dimensional conformal quantum field theory with \mathbb{Z}_n symmetry *Int. J. Mod. Phys. A* **3** 507–20
- [45] Hirota R 1981 Discrete analogue of a generalized Toda equation *J. Phys. Soc. Japan* **50** 3785–91
- [46] Miwa T 1982 On Hirota’s difference equations *Proc. Japan Acad.* **58** 9–12